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THE EFFECT OF FLOW PULSATIONS ON CORIOLIS MASS FLOW METERS

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It has been reported that the accuracy of Coriolis mass flow meters can be adversely affected by the presence of pulsations (at particular frequencies) in the flow. A full analysis of the transient performance of a commercial Coriolis meter is only possible using finite element techniques. However, this is a transient, nonlinear problem in which the space and time variables are not (strictly) separable and the finite element techniques for tackling such problems make it desirable to have an analytical solution for a simplified meter, against which the finite element solution can be compared. This paper reports such a solution. The solution will also provide guidance for experiments. Existing analytical solutions for the performance of Coriolis meters in steady flow (a complex eigenvalue problem) are not easily extended to the transient flow case. The paper thus begins with the presentation of an alternative solution for steady flow through a simple, straight tube, Coriolis meter and it is notable that this solution gives a simple analytical expression for the experimentally observed small change in the resonant frequency of the meter, with flow rate, as well as an analytical expression for the meter sensitivity. The analysis is extended to the transient case, using classical, forced vibration, modal decomposition techniques. The solution shows that, unlike the steady flow case where the detector signals contain components at the drive frequency and the second mode frequency (Coriolis frequency), for pulsatile flow the detector signals will in general contain components involving at least four frequencies. It is demonstrated that the meter error depends on the algorithm used to estimate the phase difference from the detector signals. The particular flow pulsation frequencies which could possibly lead to large meter errors are identified.

1. INTRODUCTION

CORIOLIS MASS FLOW METERS are taking an ever increasing share of the flow meter market, and the manufacturers claim that they give accurate measurements even in the presence of flow transients. Thus they are widely used in batching processes, even when the flows are driven by positive displacement pumps with consequent pulsations in the flow rate. Only one report of an investigation into the influence of flow pulsations on meter accuracy has been found. Vetter & Notzon (1994) tested two sizes of U-tube type Coriolis meters and reported large meter errors for particular pulsation frequencies. The tests included both single-frequency, sinusoidal, flow pulsations and pulsations resulting both from piston pumps with fast acting valves and from gear pumps. In the latter cases, although the basic frequencies of the pumps were quite low, the sharp waveforms led to significant Fourier components at higher frequencies, and it was found that these could cause errors. Vetter & Notzon suggest that problems arise when the frequency of the flow pulsations is equal to one of the resonant frequencies of the meter and that they are most severe when the frequency is equal to the Coriolis frequency (the use of this name for the frequency of the vibration mode responsible for the Coriolis effect is not universal, possibly because

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it is not the dominant frequency of the signals which appear at the motion detectors on a meter).

There have been a small number of reports of pulsation problems, from meter users, but in no case has it been possible to obtain quantitative information about the flow pulsations, and it has been suggested that in the majority of these cases the errors may be due to mechanical vibrations rather than flow pulsations. In one case, it was suggested that there may have been an airborne acoustic coupling between a strong sound source and the meter. The meter manufacturers have well-defined recommendations concerning the mounting of their meters and they claim that adherence to these recommendations will overcome the majority of pulsation problems. Although errors due to flow pulsations may only occur rarely, it is still important to gain an understanding of the problem, because it seems quite probable that the errors may occur without any visible indication that there is a problem. In this sense, the problem may be similar to the lock-in error problem with vortex shedding flowmeters (Mottram & Robati 1985), rather than the errors due to pulsating flow through an orifice plate, where a wildly fluctuating pressure difference gives a clear indication of a problem.

There have been a considerable number of publications dealing with the dynamics of fluid/pipe interactions in recent years. Surveys of the important work have been published by Paidoussis & Issid (1974) and by Paidoussis & Li (1993). Almost all the work has been directed towards questions of stability and where specific system information is given, typical dimensionless stiffnesses are some three orders of magnitude less than those typical of Coriolis flow meters. In both the above-referenced survey articles, and in papers such as Ariaratnam & Namachchivaya (1986), the effects of pulsatile fluid flow are considered. The latter paper concludes that 'combination resonances' are associated with sums of natural system frequencies and fluid pulsation frequencies. For typical Coriolis meter systems, which are driven at a natural frequency, it might be expected that both sums and differences of the drive and the fluid pulsation frequencies will be significant.

There are two main published reports of analytical descriptions of the steady flow behaviour of Coriolis flow meters. Sultan & Hemp (1989) describe an analysis for a U-tube meter in which the need for separate equations to describe the straight and curved parts of the tube leads to requirements for the matching of boundary conditions, involving substantial numerical computations. Raszillier & Durst (1991) report an analysis for a simpler, straight tube, meter using variational techniques which lead to an approximate analytical solution. The general approach of Raszillier & Durst was followed in the present work because the objective was to obtain an analytical understanding of the effects of pulsating flow; the complexity of the Sultan & Hemp solution suggested that even if pulsating flow effects could be included, the results were unlikely to be accessible in a simple analytic form.

The adoption of the general approach of Raszillier & Durst included the adoption of many of their simplifying approximations. Thus, the problem tackled in the present work is that of pulsating flow through a simple meter comprising a length of straight, uniform pipe, rigidly built-in at its two ends and driven at the centre at the fundamental resonant frequency. The pipe is assumed to behave as a simple Euler beam undergoing one-dimensional, transverse vibration. All effects of axial stresses arising from end loads, including the influence of the fluid pressure are neglected. The fluid is assumed to be inviscid and incompressible and to be perfectly coupled to the motion of the pipe, i.e. the fluid has no inner degrees of freedom. The physical size of typical meters, together with the expected range of flow pulsation frequencies, suggests that the meter length will never be more than 1/10th of the wavelength of the flow pulsations. A further simplification was therefore achieved by assuming that the longitudinal velocity of the fluid through the meter is a function only of time and not of position (the motion of the fluid due to the vibratory motion of the pipe is of course a function of both time and position).

In this simple model of a Coriolis flow meter, it is assumed that there are motion detectors at the 1/4 and 3/4 points and that the quantity of interest is the phase difference between the signals from these detectors. The primary objective of this work is to determine the effect of flow pulsations on the detector signals. In commercially available meters, the techniques used to determine the phase difference between the detector signals vary from manufacturer to manufacturer and are strictly confidential. Thus it is not possible, in this paper, to express the results as ranges of pulsation frequency and amplitude where the normal meter reading will be in error. A further result of the constraints of commercial confidentiality is that although all meters use a sytem of feedback from the detectors to the unit which drives the meter at a resonant frequency, it is not possible to obtain specific details of this feedback system.

2. FORMULATION OF PROBLEM AND METHOD OF SOLUTION

The transverse vibratory motion of the pipe and the fluid is represented by writing the displacement, u, as a function of the distance, x, along the pipe from one end, and of the time, t. Writing force = mass × acceleration for the fluid and recognizing that since $u = u(x, t)$, $\mathrm{d}u/\mathrm{d}t = \partial u/\partial t + (\partial u/\partial x)(\mathrm{d}x/\mathrm{d}t) = \partial u/\partial t + V(t)\partial u/\partial x$, the motion of the fluid is described by

$$m_f \frac{\partial^2 u}{\partial t^2} + 2m_f V \frac{\partial^2 u}{\partial x \partial t} + m_f \frac{\mathrm{d}V}{\mathrm{d}t}\frac{\partial u}{\partial x} + m_f V^2 \frac{\partial^2 u}{\partial x^2} = \lambda, \tag{1}$$

where m_f is the mass of fluid per length of pipe, $V \equiv V(t)$ is the longitudinal velocity of the fluid, and λ is the force per length exerted on the fluid by the constraining pipe. Similarly, the motion of the pipe is described by

$$m_p \frac{\partial^2 u}{\partial t^2} + EI \frac{\partial^4 u}{\partial x^4} = -\lambda, \tag{2}$$

where m_p is the mass of the pipe per length and E and I are, respectively, the Young's modulus and the second moment of area of the pipe.

Eliminating λ between equations (1) and (2) gives the equation of motion of the combined system,

$$(m_p + m_f)\frac{\partial^2 u}{\partial t^2} + EI \frac{\partial^4 u}{\partial x^4} + m_f\left[2V \frac{\partial^2 u}{\partial x \partial t} + \frac{\mathrm{d}V}{\mathrm{d}t}\frac{\partial u}{\partial x} + V^2 \frac{\partial^2 u}{\partial x^2}\right] = 0. \tag{3}$$

For a meter of length L the boundary conditions with respect to x are

$$u(0, t) = u(L, t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(L, t) = 0,$$

and it is not necessary to specify the boundary conditions with respect to t at this point.

Equation (3) differs from the corresponding equation for steady flow, which was derived by Raszillier & Durst, among others, by the presence of the second term in the [ ]. The presence of this extra term is consistent with the assumed neglect of axial tension, etc., but it should be noted that Paidoussis & Issid have shown that when such terms are included, this term no longer appears in the same form. However, the effect of incorporating a time-dependent fluid velocity ($V = V(t)$) is much greater than is implied by the presence of this extra term. When V is a constant we have a complex eigenvalue problem, with the $2V$ term leading to the complexity and the V^2 term leading to a small change in the system stiffness (and hence to a small change in the system resonant frequencies). When $V = V(t)$, the problem changes from being a complex eigenvalue problem to a problem in which the space and time variables are not separable. This is emphasized by expanding the term in square brackets in equation (3) for the case $V = V_0[1 + \alpha \sin(\omega_f t)]$, obtaining

$$[\ ] = 2V_0 \frac{\partial^2 u}{\partial x \partial t} + V_0^2 \frac{\partial^2 u}{\partial x^2} + 2V_0 \alpha \sin(\omega_f t) \frac{\partial^2 u}{\partial x \partial t} + V_0 \alpha \omega_f \cos(\omega_f t) \frac{\partial u}{\partial x}$$

$$+ 2V_0^2 \alpha \sin(\omega_f t) \frac{\partial^2 u}{\partial x^2} + V_0^2 \alpha^2 \sin(\omega_f t) \frac{\partial^2 u}{\partial x^2} \quad . \tag{4}$$

Values of α which are of interest are in the range $0 < \alpha < 1$, so that for the general unsteady case, in addition to two terms like those considered for steady flow by Raszillier & Durst, there are four terms which imply a solution in which the dependencies on space and time are not separable. It is clear that a simple form of general analytical solution of equation (3) is not possible for velocity distributions of the type $V = V_0[1 + \alpha \sin(\omega_f t)]$.

However, when the orders of magnitude of the terms in equation (4) are evaluated, for conditions typical of Coriolis flow meters, it is found that they are much smaller than the orders of magnitude of the first two terms in equation (3). Thus, the problem is still similar to that solved by Raszillier & Durst, in the sense that the first two terms in equation (3) will have a dominant influence on the solution. Thus, it is reasonable to assume a solution of the form

$$u(x, t) = \sum_{n=1}^{\infty} W_n(x)\, q_n(t),$$

where the $W_n(x)$ are the mode shapes, obtained from the solution to the equation formed by setting the first two terms in equation (3) equal to zero, and the $q_n(t)$ are usually referred to as generalized coordinates. Furthermore, the work of Raszillier & Durst suggests that it should not be necessary to continue the summation beyond the first two or three terms.

For the present boundary conditions, the mode shapes are given by

$$W_n(x) = \sinh(\beta_n x) - \sin(\beta_n x) + \alpha_n[\cosh(\beta_n x) - \cos(\beta_n x)],$$

where $\alpha_n = [\sinh(\beta_n L) - \sin(\beta_n L)]/[\cos(\beta_n L) - \cosh(\beta_n L)]$ and the $\beta_n L$ are solutions to $\cos(\beta_n L)\cosh(\beta_n L) = 1$. For the case of $V = 0$, the generalized coordinates are given by $q_n(t) = \sin(\omega_n t)$, where $\omega_n = (\beta_n L)^2[EI/L^4(m_p + m_f)]^{1/2}$.

When the assumed form of solution is substituted into equation (3), after some re-arrangement, the equation can be written as

$$0 = \sum_{n=1}^{\infty} \omega_n{}^2 W_n(x)\, q_n(t) + \sum_{n=1}^{\infty} W_n(x) \frac{\mathrm{d}^2 q_n(t)}{\mathrm{d}t^2}$$

$$+ \frac{m_f}{(m_p + m_f)} \left[2V \sum_{n=1}^{\infty} \frac{\mathrm{d}W_n(x)}{\mathrm{d}x} \frac{\mathrm{d}q_n(t)}{\mathrm{d}t} + \frac{\mathrm{d}V}{\mathrm{d}t} \sum_{n=1}^{\infty} \frac{\mathrm{d}W_n(x)}{\mathrm{d}x} q_n(t) + V^2 \sum_{n=1}^{\infty} \frac{\mathrm{d}^2 W_n(x)}{\mathrm{d}x^2} q_n(t) \right]. \tag{5}$$

Multiplying equation (5) through by the general mode shape $W_m(x)$, integrating with respect to x from $x = 0$ to $x = L$ and imposing the condition of orthogonality of normal modes gives, for mode m,

$$\frac{d^2 q_m(t)}{dt^2} + \omega_m{}^2 q_m(t) + \frac{m_f}{(m_p + m_f)} \frac{1}{\int_0^L W_m^2(x)\,dx} \left[2V \sum_{n=1}^{\infty} \left\{ \frac{dq_n(t)}{dt} \int_0^L W_m(x) \frac{dW_n(x)}{dx} dx \right\} \right.$$

$$\left. + \frac{dV}{dt} \sum_{n=1}^{\infty} \left\{ q_n(t) \int_0^L W_m(x) \frac{dW_n(x)}{dx} dx \right\} + V^2 \sum_{n=1}^{\infty} \left\{ q_n(t) \int_0^L W_m(x) \frac{d^2 W_n(x)}{dx^2} dx \right\} \right] = 0. \quad (6)$$

Equation (6) describes an infinite set of coupled equations for the generalized coordinates. The following coefficients can be defined in terms of the mode shape integrals which appear in equation (6):

$$\theta_m = \frac{1}{L} \int_0^L W_m^2(x)\,dx,$$

$$\psi_{m,n} = \int_0^L W_m(x) \frac{dW_n(x)}{dx} dx,$$

$$\chi_{m,n} = L \int_0^L W_m(x) \frac{d^2 W_n(x)}{dx^2} dx.$$

These coefficients have been evaluated up to $m = n = 6$ and are given in Table 1. (All the zero values in Table 1 and the identities $\psi_{m,n} = -\psi_{n,m}$ and $\chi_{m,n} = \chi_{n,m}$ are the result of mathematical analysis as well as numerical evaluation). However, the work of Raszillier & Durst (1991) and the extensions to that work by Raszillier, Alleborn & Durst (1993), suggest that a good approximation can be obtained by considering only the first two modes. Introducing this approximation, equation (6) yields the following pair of equations for the generalized coordinates q_1 and q_2 (in which the explicit designation of the dependent variable has been dropped and terms which are identically zero have been omitted) :

$$\frac{d^2 q_1}{dt^2} + \omega_1{}^2 q_1 - \frac{m_f}{L\theta_1(m_p + m_f)} \left[2\psi_{1,2} V \frac{dq_2}{dt} + \psi_{1,2} \frac{dV}{dt} q_2 + \frac{V^2}{L} \chi_{1,1} q_1 \right] = 0, \quad (7)$$

$$\frac{d^2 q_2}{dt^2} + \omega_2{}^2 q_2 - \frac{m_f}{L\theta_2(m_p + m_f)} \left[2\psi_{2,1} V \frac{dq_1}{dt} + \psi_{2,1} \frac{dV}{dt} q_1 - \frac{V^2}{L} \chi_{2,2} q_2 \right] = 0. \quad (8)$$

A study of the results of Raszillier & Durst suggests that for all practical Coriolis meters, q_2 is between 100 and 1000 times smaller than q_1.

Although the behaviour of a simple Coriolis meter in steady flow has been described by Raszillier & Durst (1991) and can be inferred from the work of Sultan & Hemp (1989) for a U-tube meter, it is appropriate to demonstrate that the somewhat different approach employed in the present work gives comparable results.

TABLE 1

Numerical values of the mode shape integrals in equation (6)

			$\psi_{m,n}$					
m	θ_m	n	1	2	3	4	5	6
1	1·0359		0	3·399	0	− 0·923	0	− 0·438
2	0·9984		3·399	0	− 5·512	0	− 1·725	0
3	1·0000		0	5·512	0	− 7·635	0	− 2·528
4	1·0000		0·923	0	7·635	0	− 9·704	0
5	1·0000		0	1·725	0	9·704	0	− 11·752
6	1·0000		0·438	0	2·528	0	11·752	0

		$\chi_{m,n}$					
m	n	1	2	3	4	5	6
1		− 12·74	0	9·90	0	7·75	0
2		0	− 45·98	0	17·13	0	15·17
3		9·90	0	− 98·91	0	24·35	0
4		0	17·13	0	− 171·58	0	31·26
5		7·75	0	24·35	0	− 264·00	0
6		0	15·17	0	31·26	0	− 376·15

3. STEADY FLOW THROUGH A CORIOLIS METER

For steady flow, equation (7) can be written as

$$\frac{\mathrm{d}^2 q_1}{\mathrm{d}t^2} + q_1\left[\omega_1{}^2 - \frac{\chi_{1,1} m_f V^2}{L^2\theta_1(m_p + m_f)}\right] = \frac{2\psi_{1,2} V m_f}{L\theta_1(m_p + m_f)}\frac{\mathrm{d}q_2}{\mathrm{d}t}. \tag{9}$$

The solution to equation (9) is of the form

$$q_1 = C_{1,0}\sin(\gamma_1 t) + C_{1,1}\cos(\gamma_1 t) + \{\text{particular integral}\}, \tag{10}$$

where

$$\gamma_1 = \sqrt{\omega_1{}^2 - \frac{\chi_{1,1} m_f V^2}{L^2\theta_1(m_p + m_f)}}. \tag{11}$$

Equation (11) shows how the effective frequency of the first mode is changed very slightly (typically 0·1–0·2%) by the influence of a steady flow through the meter. This effect is well known both from the more general work on the dynamics of fluid/pipe interactions [e.g., Paidoussis & Li (1993)] and from the computed results of the analysis by Sultan & Hemp (1989), as well as from a finite element study reported by Stack, Garnett & Pawlas (1993). However, in none of these sources is there a simple expression which can be evaluated for typical meter parameters, in the way that is possible from equation (11). Figure 1 shows a comparison of the predictions of equation (11) with the finite element and experimental results reported by Stack *et al*. Because of the influence of different meter geometries, the comparison is made in terms of the 'normalized frequency' [frequency/(frequency for zero flow)] against the flow rate expressed as a percentage of the normal maximum flow rate for the meter. In a finite element study, the results of which are not yet ready for publication,

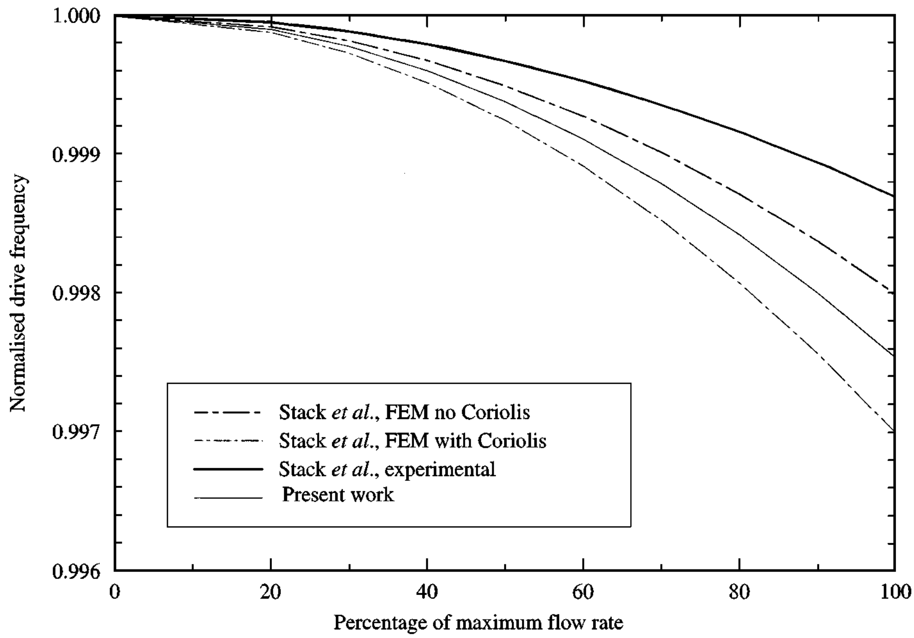

Figure 1. Variation of the drive frequency with flow rate: present predictions compared to smooth curve representations of the data from figure 11 of Stack *et al.* (1993).

computations using exactly the same approximations as in the work reported here, gave results in very close agreement with equation (11).

Before examining the significance of the particular integral in equation (10) it is convenient to examine the equation for the second mode. For steady flow, equation (7) can be written as

$$\frac{\mathrm{d}^2 q_2}{\mathrm{d}t^2} + q_2\left(\omega_2{}^2 - \frac{\chi_{2,2} m_f V^2}{L^2\theta_2(m_p + m_f)}\right) = \frac{2\psi_{2,1} V m_f}{L\theta_2(m_p + m_f)}\frac{\mathrm{d}q_1}{\mathrm{d}t}. \tag{12}$$

The solution to equation (12) is of the form

$$q_2 = C_{2,0}\sin(\gamma_2 t) + C_{2,1}\cos(\gamma_2 t) + \{\text{particular integral}\}, \tag{13}$$

where

$$\gamma_2 = \sqrt{\omega_2{}^2 - \frac{\chi_{2,2} m_f V^2}{L^2\theta_2(m_p + m_f)}}. \tag{14}$$

Equation (14) shows that there is a small change in the effective frequency of the second mode, with flow rate, similar to that found for the first mode.

The model meter considered here does not have any damping or any driving force. A real meter of this geometry would be driven, at the centre of its span, by a signal derived from one (or both) of the detectors. The feedback system would be arranged to keep a constant amplitude (first mode) motion at the detectors. Although meter manufacturers will not release details of the feedback process, it is clear that the drive mechanism will tend to emphasise the significance of the $\sin(\gamma_1 t)$ and the $\cos(\gamma_1 t)$ terms in equation (10) relative to the particular integral. Since the origin of the time scale is arbitrary, it is appropriate to choose it so that $C_{1,1} = 0$. The work of Raszillier & Durst suggests that the contribution of the particular integral to equation (10) will be of the order of $10^{-3}C_{1,0}$. This was confirmed by a successive approximation procedure in which the particular integral in equation (10) was first taken to be zero, q_2 was then determined from equation (12), using the approximated q_1, and then equation (9) was re-solved, evaluating the particular integral using the derived form of q_2. For typical meter parameters the 'improved' solution for q_1 differed from the first estimate by approximately 0·1%. Thus, it is reasonable to take q_1 as $C_{1,0}\sin(\gamma_1 t)$ and equation (13) becomes

$$q_2 = C_{2,0}\sin(\gamma_2 t) + C_{2,1}\cos(\gamma_2 t) + \frac{2\psi_{2,1}C_{1,0}V\gamma_1 m_f}{L\theta_2(m_p + m_f)}\frac{\cos(\gamma_1 t)}{(\gamma_1^2 - \gamma_2^2)}. \tag{15}$$

In considering the evaluation of $C_{2,0}$ and $C_{2,1}$ it is necessary to consider the starting conditions carefully. There are two main possibilities: either to assume that the fluid is flowing steadily with the meter not driven and then the drive is suddenly switched on; or the meter is being driven with zero flow and then the flow is suddenly started. The present model does not allow the simulation of the first of these conditions and it is the second which occurs more usually in practice, since it is usual to check (and adjust if necessary) the meter output at zero flow. For a meter driven at zero flow, q_2 will be identically zero. When the flow is suddenly started from this condition, equation (15) becomes (neglecting the small difference between γ_1 and ω_1)

$$q_2 = \frac{2\psi_{2,1}C_{1,0}V\gamma_1 m_f}{L\theta_2(m_p + m_f)(\gamma_1^2 - \gamma_2^2)}\left[\cos(\gamma_1 t) - \cos(\gamma_2 t)\right]. \tag{16}$$

The essential features of equation (16) are confirmed by observations of the detector signals on actual meters. When the meters are started in the way which is modelled above, the detector signals always contain a small component at a frequency γ_2, in addition to the dominant signal at the drive frequency γ_1.

Combining the mode shapes and the generalized coordinates, within the approximations of the present solution, the motion of the meter is described by

$$u(x,t) = C_{1,0}\left[W_1(x)\sin(\gamma_1 t) + \frac{2\psi_{2,1}W_2(x)\;V\gamma_1 m_f}{L\theta_2(M_p + m_f)\;(\gamma_1^2 - \gamma_2^2)}\{\cos(\gamma_1 t) - \cos(\gamma_2 t)\}\right]. \tag{17}$$

It is often claimed that the output signal from a Coriolis meter is derived from the phase difference between the signals from the two detectors, which for the present model are assumed to be placed at $x = L/4$ and $x = 3L/4$. When, as demonstrated in equation (17), the detector signals contain components at frequencies other than the drive frequency, it is the phase difference between those components of the detector signals which are at the drive frequency which is proportional to the mass flow rate. Substituting the values of $W_1(x)$ and $W_2(x)$ at the detector points, the phase difference, Φ, between the γ_1 components of the two detector signals, is given from equation (17) by

$$\Phi = \frac{2\psi_{2,1}\gamma_1 \dot{m}}{L\theta_2(m_p + m_f)\;(\gamma_1^2 - \gamma_2^2)}\left\{\frac{W_2(L/4)}{W_1(L/4)} - \frac{W_2(3L/4)}{W_1(3L/4)}\right\}, \tag{18}$$

where $\dot{m}(= Vm_f)$ is the mass flow rate of the fluid through the meter.

Since $W_1(L/4) = W_1(3L/4)$ and, for a typical meter, $(\gamma_1^2 - \gamma_2^2)$ differs from $(\omega_1^2 - \omega_2^2)$ by less than 10^{-5}, the definition of ω can be introduced so that equation (18) can be re-arranged as,

$$\dot{m} = \frac{EI[(\beta_1 L)^4 - (\beta_2 L)^4]\theta_2}{2\psi_{2,1}L^3} \frac{W_1(L/4)}{W_2(L/4) - W_2(3L/4)} \Delta t, \tag{19}$$

where $\Delta t(= \Phi/\gamma_1)$ is the time difference between the detector signals.

Equation (19) and the derivation leading to it are not new. They have been introduced for completeness and to demonstrate a parameter dependence identity with the result quoted by Raszillier & Durst.

4. PULSATING FLOW THROUGH A CORIOLIS METER

For a time-dependent velocity of the form $V = V_0[1 + \alpha \sin(\omega_f t)]$, equation (7) can be written as

$$\frac{d^2 q_1}{dt^2} + q_1\left(\omega_1{}^2 - \frac{\chi_{1,1} m_f V_0{}^2}{L^2\theta_1(m_p + m_f)}\right) = \frac{d^2 q_1}{dt^2} + \gamma_1^2 q_1$$

$$= \frac{\psi_{1,2} m_f V_0}{L\theta_1(m_p + m_f)}\left[2\{1 + \alpha \sin(\omega_f t)\}\frac{dq_2}{dt} + \alpha\omega_f \cos(\omega_f t)\, q_2 + \frac{\chi_{1,1} V_0}{\psi_{1,2} L} 2\alpha \sin(\omega_f t)\, q_1\right], \tag{20}$$

where the justification for the omission of terms involving $V_0^2 q_2$ is the same as in the derivation of equation (9), and the term involving $\alpha^2 V_0^2 \sin^2(\omega_f t)$ is neglected because practically occurring values of α are in the range $0 \leq \alpha \leq 0{\cdot}2$.

Equation (20) shows that the effective frequency of the first mode is changed slightly by the mean velocity in exactly the same way as for steady flow [see equation (11)]. The last term in equation (20) apparently describes a small periodic variation in the effective system stiffness (for the first mode) and it is tempting to write the equation as

$$\frac{d^2 q_1}{dt^2} + \gamma_1{}^2 q_1 - \frac{2\alpha\chi_{1,1} m_f V_0^2}{L^2\theta_1(m_p + m_f)} \sin(\omega_f t)\, q_1$$

$$= \frac{\psi_{1,2} m_f V_0}{L\theta_1(m_p + m_f)}\left[2\{1 + \alpha \sin(\omega_f t)\}\frac{dq_2}{dt} + \alpha\omega_f \cos(\omega_f t)\, q_2\right]. \tag{21}$$

The complementary function part of the solution to equation (21) is the solution to a Mathieu equation and this would suggest the possibility of instabilities for $\omega_f = \gamma_1$ and $\omega_f = 2\gamma_1$. However, before considering this suggestion of instability further, it must be noted that practical Coriolis flow meters are driven at a modal frequency (in the present case γ_1) by feedback using a signal derived from the motion detectors. Meter manufacturers will not release exact details of their feedback systems, but those manufacturers who were prepared to comment on this matter all agreed that the drive would not follow changes occuring during a time of the order of $2\pi/\omega_f$. Thus, a practical Coriolis meter is effectively driven at a constant frequency γ_1, and it is not clear what the effect of the periodic variation of the effective system stiffness, contained in equation (21), will be. With this in mind, consideration of the effects of fluid pulsations will be continued with the generalized coordinate for the first mode described by equation (20) rather than by equation (21).

The problems regarding the possible influence of the feedback and the meter drive do not have a significant effect on the equation describing the generalized coordinate for the second mode, which, by analogy with the derivation of equation (21), can be written as

$$\frac{\mathrm{d}^2 q_2}{\mathrm{d}t^2} + {\gamma_2}^2 q_2 - \frac{2\alpha\chi_{2,2} m_f V_0^2}{L^2\theta_2(m_p + m_f)}\sin(\omega_f t)\, q_2$$

$$= \frac{\psi_{2,1} m_f V_0}{L\theta_2(m_p + m_f)}\left[2\{1 + \alpha\sin(\omega_f t)\}\frac{\mathrm{d}q_1}{\mathrm{d}t} + \alpha\omega_f\cos(\omega_f t)\, q_1\right], \tag{22}$$

where γ_2 is defined by equation (14), except that V is replaced by V_0.

In order to simplify the discussion of the solution to equations (20) and (22), it is convenient to define the following dimensionless variables:

$$\varepsilon_m = \frac{2\alpha\chi_{m,m} m_f {V_0}^2}{L^2\theta_m(m_p + m_f)\ \gamma_m^2} \tag{23}$$

and

$$\zeta_{m,n} = \frac{\psi_{m,n} m_f V_0 \omega_f}{L\theta_m(m_p + m_f)}. \tag{24}$$

In terms of the new variables, equations (20) and (22) become

$$\frac{\mathrm{d}^2 q_1}{\mathrm{d}t^2} + {\gamma_1}^2 q_1 = \zeta_{1,2}\left[\frac{2}{\omega_f}\{1 + \alpha\sin(\omega_f t)\}\frac{\mathrm{d}q_2}{\mathrm{d}t} + \alpha\cos(\omega_f t)\, q_2\right] + \varepsilon_1 {\gamma_1}^2 \sin(\omega_f t)\, q_1 \tag{25}$$

and

$$\frac{\mathrm{d}^2 q_2}{\mathrm{d}t^2} + \gamma_2^2\{1 - \varepsilon_2\sin(\omega_f t)\} q_2 = \zeta_{2,1}\left[\frac{2}{\omega_f}\{1 + \alpha\sin(\omega_f t)\}\frac{\mathrm{d}q_1}{\mathrm{d}t} + \alpha\cos(\omega_f t)\, q_1\right]. \tag{26}$$

Adopting the same approach as for the case of steady flow, it will be assumed that the meter starts from a condition of being filled with liquid and driven in pure first mode motion so that $u(x, t) = W_1(x)\sin(\omega_1 t)$. At time $t = 0$ the pulsatile flow is suddenly started. Examination of the solutions to equations (25) and (26) shows that there are wide ranges of values for ω_f for which q_2 is of the order of 0.1% of q_1, so that, for these conditions, the solution to equation (25) can be approximated as $q_1 = C_{1,0}\sin(\gamma_1 t)$ (as in the case of steady flow). The solution for pulsating flow, which will be developed from this approximation for q_1, will enable the identification of ranges of ω_f where the approximation is likely to fail. Because these ranges are small and because, for reasons noted earlier, it is not possible to derive estimates of meter error, detailed solutions for cases where q_2 is large will not be attempted.

When $q_1 = C_{1,0}\sin(\gamma_1 t)$ is substituted into equation (26), the result can be written as

$$\frac{\mathrm{d}^2 q_2}{\mathrm{d}t^2} + \gamma_2^2\{1 - \varepsilon_2\sin(\omega_f t)\} q_2$$

$$= C_{1,0}\zeta_{2,1}\left[\frac{2\gamma_1}{\omega_f}\cos(\gamma_1 t) + \alpha\left\{\frac{\omega_f - 2\gamma_1}{2\omega_f}\sin[(\gamma_1 - \omega_f)\, t] + \frac{\omega_f + 2\gamma_1}{2\omega_f}\sin[(\gamma_1 + \omega_f) t]\right\}\right]. \tag{27}$$

The solution to equation (27) is of the form {complementary function} + {particular integral}, where the complementary function is the solution to the equation formed by setting the left-hand side (l.h.s.) of equation (27) equal to zero. The complementary equation for equation (27) is the solution to a Mathieu equation [as was previously noted for equation (21)]. This solution has regions of instability centred on $\omega_f = \gamma_2$ and on $\omega_f = 2\gamma_2$,

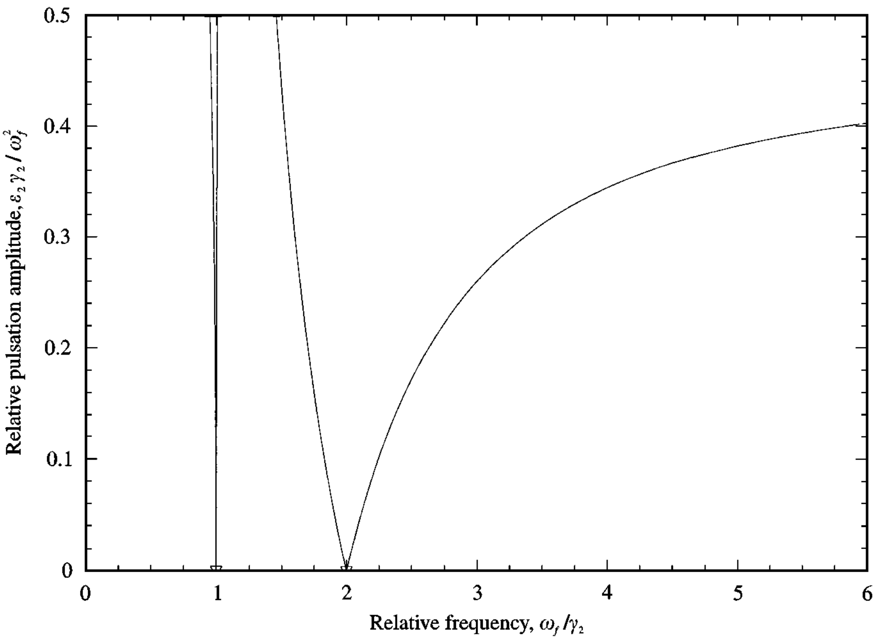

Figure 2. Regions of unstable solution of the Mathieu equation (plotted from data in Rao (1995).

the approximate widths of the regions being $\varepsilon_2^2\gamma_2/2$ and $\varepsilon_2\gamma_2$ respectively. A graphical representation of the regions of instability is shown in Figure 2, which is reproduced from the analysis presented by Rao (1995). In assessing the significance of these regions of instability it should be noted that, for a typical straight-tube Coriolis meter, ε_2 is of the order of 10^{-4}. In an engineering environment, it is inconceivable that any fluid pulsation frequency could be sufficiently steady and noise-free to remain within the instability region centered on $\omega_f = \gamma_2$.

Before leaving consideration of these regions of instability, it is appropriate to note that the existence of such regions is demonstrated in all the general treatments of fluid-pipe interactions which are discussed in the Introduction. In these treatments [e.g., Paidoussis & Issid (1974)] the region centred on $\omega_f = 2\gamma_2$ is referred to as the principal primary region and that centred on $\omega_f = \gamma_2$ is referred to as the principal secondary region. It is notable that the region referred to as the second primary region centred on $\omega_f = 2\gamma_2/3$ does not appear in the present analysis. Figure 9 in Paidoussis & Issid suggests that the second primary region is of even narrower extent than the principal secondary region, so that for the reasons disscussed above it can have no practical significance for Coriolis meters.

Outside the regions of instability the influence of the ε_2 term in equation (27) will be very small and it will not significantly affect the performance of the meter. The contribution of this term to the complementary function will therefore be neglected in the stable regions. The particular integrals arising from the right-hand side (r.h.s.) of equation (27) are

$$C_{1,0}\zeta_{2,1}\left\{-\frac{2\gamma_1}{\omega_f(\gamma_2^2-\gamma_1^2)}\cos(\gamma_1 t)+\mathcal{O}\left[\frac{\varepsilon_2}{(\gamma_2^2-\gamma_1^2)}\right]\right\},$$

$$C_{1,0}\zeta_{2,1}\alpha\left\{\frac{\omega_f-2\gamma_1}{\omega_f[\gamma_2^2-(\gamma_1-\omega_f)^2]}\sin[(\gamma_1-\omega_f)\,t]+\mathcal{O}\left[\frac{\varepsilon_2}{[\gamma_2^2-(\gamma_1-\omega_f)^2]}\right]\right\}$$

and

$$C_{1,0}\zeta_{2,1}\alpha\left\{\frac{\omega_f + 2\gamma_1}{\omega_f[\gamma_2^2 - (\gamma_1 + \omega_f)^2]}\sin[(\gamma_1 + \omega_f)\, t] + \mathcal{O}\left[\frac{\varepsilon_2}{[\gamma_2^2 - (\gamma_1 + \omega_f)^2]}\right]\right\}.$$

Since ε_2 is of the order of 10^{-4}, the terms $\mathcal{O}(\varepsilon_2)$ are negligible compared to the principal parts of the particular integrals except for the special case of $\omega_f \simeq 2\gamma_1$. When the constants of integration in the particular integral are evaluated from the starting conditions in the same way as for steady flow, the complete solution for the generalized coordinate for the second mode can be written, except in the regions of instability, as

$$q_2 = C_{1,0}\zeta_{2,1}\left(\frac{2\gamma_1}{\omega_f(\gamma_1^2 - \gamma_2^2)}\,[\cos(\gamma_1 t) - \cos(\gamma_2 t)]\right.$$
$$\left. + \alpha\left\{\frac{\omega_f - 2\gamma_1}{\omega_f[{\gamma_2}^2 - (\gamma_1 - \omega_f)^2]}\sin[(\gamma_1 - \omega_f)\,t] + \frac{\omega_f + 2\gamma_1}{\omega_f[\gamma_2^2 - (\gamma_1 + \omega_f)^2]}\sin[(\gamma_1 + \omega_f)\,t]\right\}\right). \tag{28}$$

Thus, the motion seen by the detectors is

$$C_{1,0}\left\{W_1(L_d)\sin(\gamma_1 t) + W_2(L_d)\,\frac{2\gamma_1}{\omega_f(\gamma_1^2 - \gamma_2^2)}\,[\cos(\gamma_1 t) - \cos(\gamma_2 t)]\right\}$$
$$+ C_{1,0}W_2(L_d)\alpha\left\{\frac{\omega_f - 2\gamma_1}{\omega_f[\gamma_2^2 - (\gamma_1 - \omega_f)^2]}\sin[(\gamma_1 - \omega_f)\,t]\right.$$
$$\left. + \frac{\omega_f + 2\gamma_1}{\omega_f[\gamma_2^2 - (\gamma_1 + \omega_f)^2]}\sin[(\gamma_1 + \omega_f)\, t]\right\}, \tag{29}$$

where L_d is $L/4$ or $3L/4$, depending on which detector is being considered.

Equation (29) shows that, in the presence of a pulsating flow, the signals from the detectors contain the same information as for a steady flow at the mean flow rate, but in addition they also contain information at frequencies which are the sum and difference of the first (driven) mode frequency and the flow pulsation frequency. Unlike the steady flow case where the detector signals contained only information at the first and second mode frequencies which are quite separate, for the pulsating flow case the sum or difference frequencies could be close to (or even equal to) one of the mode frequencies. This feature is significant because, although manufacturers will not release details of their methods for determining the time difference (or phase difference and signal frequency) between the detector signals, almost all potential methods will give problems when there is a beat frequency of the order of 1 Hz.

Examination of the flow pulsation frequencies which are likely to cause problems shows that for $\omega_f = \gamma_1 + \gamma_2$ or $\omega_f = \gamma_2 - \gamma_1$, equation (29) predicts infinite amplitude motion, corresponding to the resonant frequency, γ_2 being excited by the pulsations. The significance of this effect in a real meter would depend on the extent to which the excitation of the resonant frequency was limited by internal damping. For $\omega_f = 2\gamma_2$ there is the possibility of the pulsations exciting the Mathieu instability (parametric resonance). For $\omega_f = 2\gamma_1$ there are two points to be taken into account, firstly the possibility of a Mathieu instability (but see earlier comment about the possible influence of sytem feedback) and secondly, the neglect of higher-order terms in the evaluation of the particular integrals leading to equation (29) is not a valid approximation. Finally, for ω_f close to (but not equal to) $2\gamma_1$ there will be problems of a very low beat frequency, with the consequences referred to above.

The foregoing conclusions regarding the behaviour of a Coriolis meter in a pulsatile flow can be compared to the experimental results reported by Vetter & Notzon (1994). Unfortunately, there is no mention in their paper of the method which is used to determine the phase difference in their test meter, nor is there any report of the characteristics of the raw detector signals. However, the error for $\omega_f = \gamma_2$ is very prominent in their results, and their excitation did not go up to the case $\omega_f = 2\gamma_2$. The case $\omega_f = \gamma_2 - \gamma_1$ is within the span of their data, but their figure 6 shows data points at 50 and 60 Hz which are sufficiently far away from the required 52·5 Hz for the error not to show. The degree of agreement between the present analysis for a straight tube meter and the data of Vetter and Notzon for a U-tube meter is not sufficient to indicate whether the general results of the present work are generic to a range of meter shapes.

5. FLOW THROUGH AN UNDRIVEN CORIOLIS METER

In their paper, Vetter & Notzon (1994) report the determination of the characteristic resonant frequencies of their test meter by passing a pulsatile flow through the meter with the feedback and drive to the meter disconnected. Because the fluid forces are distributed along the meter, it is not immediately obvious that a pulsatile flow at a particular frequency will excite motion of the meter at that frequency. It is therefore of interest to examine this situation in terms of the theory developed above.

It must be assumed that, for any test, there will be some small motion of the meter (much less than would exist if the drive was activated) most probably transmitted, by material vibration of the connecting pipework, from the device producing the pulsatile flow. In the presence of the pulsatile flow the motion of the meter will be described by equation (3), and via the postulation of a series solution, by equation (6). The constraints on the contributions of the different modes no longer apply, and a set of equations, one for each mode, could be derived from equation (6). These equations would be similar to equations (21) and (22), but there would no longer be a good argument for resricting attention to only the first two modes. Adopting the notation used in the analysis of the driven meter, the equations for the generalized coordinates could be written as

$$\frac{d^2 q_1}{dt^2} + \gamma_1^2[1 - \varepsilon_1 \sin(\omega_f t)] = \Phi(q_2, q_3, \dots), \tag{30}$$

$$\frac{d^2 q_2}{dt^2} + \gamma_2^2[1 - \varepsilon_2 \sin(\omega_f t)] = \Phi(q_1, q_3, \dots), \tag{31}$$

$$\frac{d^2 q_3}{dt^2} + \gamma_3^2[1 - \varepsilon_3 \sin(\omega_f t)] = \Phi(q_1, q_2, \dots), \tag{32}$$

etc.

For each of the modes, when the fluid pulsation frequency is not equal to the appropriate modal resonant frequency (modified for the effect of the mean flow), very little coherent motion will be created by the fluid pulsations. However, when the pulsation frequency is equal to twice that frequency, the Mathieu instability could possibly lead to a motion at the modal resonant frequency which would grow with time until it is bounded by material damping within the structure of the meter. This possibility is constrained by the narrowness (in the frequency domain) of the regions of Mathieu instability, as explained above.

There is no indication in the work of Vetter & Notzon (1994) that they were aware of the possibility of a resonance being excited by a fluid pulsation at double the frequency, and it must be noted that there is no indication on their figure 2 of such an effect at 155 Hz, where it might have been expected. It is not clear whether this is due to geometric differences between a straight tube meter and a U-tube meter, or to coarse spacing of excitation frequencies.

6. CONCLUSIONS

A new analysis of the behaviour of Coriolis mass flow meters, using modal decomposition techniques, has been presented. For steady flow, the results confirm the dependence of the meter calibration factor on the modal resonant frequencies, first reported by Raszillier & Durst (1991). In addition the analysis provides a simple analytical expression for the small change in the resonant frequencies with flow rate which had previously been reported from experiments and from finite element analysis. For a pulsatile flow the analysis shows that, in general, the detector signals contain components at four different frequencies so that the effect on the meter calibration factor depends on the details of the method used to determine the phase difference between the two detector signals. The analysis does not provide a complete explanation of the sensitivity of the meter output to flow pulsations at the second resonant mode, as reported by Vetter & Notzon (1994).

Perhaps the most important result of this analysis is that it proves that, except for two or three discrete frequencies, a measurement of the true mean mass-flow rate, in the presence of flow pulsations, could be derived from a Coriolis mass flowmeter, given suitable processing of the detector signals. Furthermore, it would be relatively easy to derive a warning signal from the detector signals when the problem pulsation frequencies are present.

The analysis is also used to demonstrate a possible mechanism by which a pulsatile flow through an undriven meter could excite the modal resonant frequencies of the meter.

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